2-1 Tangent Line Problem

**HW 2-1:**

pp. 103-106

#17, 19, 26, 29, 30, 34, 35, 39-42 all, 64, 97
Tangent Line Problem
Slope of the Secant Line

\[ f(x_0 + \Delta x) \]

\[ f(x_0) \]

\[ x_0 \]

\[ x_0 + \Delta x \]
Derivative as Slope of the Tangent Line

\[ f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \]
Derivative Definition & Notation

- **Definition**
  - A derivative is a limit. We evaluate the slope of a secant as the distance between the secant endpoints shrinks to zero.

- **Notation**
  - The first derivative is written as:
    - $f'(x)$
    - $y'$
    - $\frac{d}{dx}[f(x)]$
    - $\frac{dy}{dx}$

  - $f'(a)$ is the value of the first derivative at $x = a$
General Derivative

Use as a formula to find the slope of any tangent line along the curve

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
Derivative at a Specific Point

If we want to know the derivative at a particular point \((a, f(a))\) use this formula:

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]
Calculate $f'(x)$

**Ex. #1**

Given $f(x) = x^2$
Calculate $f'(x)$

Ex. #2  Given  $f(x) = x^2 - 3x + 2$
Calculate $f'(4)$

**Ex. #2**

Given $f(x) = x^2 - 3x + 2$
Tangent Line & a Normal Line

A **normal line** is a line perpendicular to the tangent line.
Tangent Line & a Normal Line

Ex. #4  If \( f(x) = x^2 \), find the equation of the tangent line and normal line at \( x = 4 \)
When Does the Derivative NOT Exist?

Since derivatives are limits, we can only find the derivative at a point if BOTH SIDES of the point exist and agree.

1. Where the Graph is discontinuous like BREAK POINTS

2. Start Points for Radical Functions

3. Cusps or Corners

4. Where there are Vertical Tangents
Derivative is the Slope of the Tangent Line!

Determine the derivative at each point:

1) $f'(-0.5) =$  
2) $f'(0.8) =$  
3) $f'(1.4) =$  
4) $f'(2.6) =$  
5) $f'(3.99) =$  
6) $f'(4.001) =$  
7) $f'(100) =$  
8) $f'(3) =$  
9) $f'(2) =$
2-2 Basic Differentiation Rules and Notation

HW 2-2
pp. 115-116
#1, 3-51 EOO, 55, 57, 59, 60, 65
Goal

Learn methods to determine the derivative of a function that are more efficient than actually evaluating the limit of a difference quotient

Slope of the Tangent Line:

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
Power Rule

Given $f(x) = ax^n$ then $f'(x) = n \cdot ax^{n-1}$

**Ex. #1** Determine the first Derivative.

a) $y = 4x^3$  
   b) $f(x) = -5x^9$
Constant Rule

Given \( f(x) = c \), where \( c \) is some real number,
Then \( f'(x) = 0 \).

**Ex. #2** Find the first derivative

a) \( f(x) = 12 \)  
b) \( y = -36.25 \)  
c) \( \frac{d}{dx} \left[ \frac{2}{3} \right] \)
Sum and Difference Rule

\[ \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \]

**Ex. #3** Find the first derivative

a) \( f(x) = -3x^2 - 4x + 1 \) \quad b) \( y = \frac{5}{x^4} - \frac{2}{x^3} + \frac{x^3}{6} - 13 \)
## Derivative of Sine and Cosine

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = a \sin x )</td>
<td>( f'(x) = a \cos x )</td>
</tr>
<tr>
<td>( f(x) = b \cos x )</td>
<td>( f'(x) = -b \sin x )</td>
</tr>
</tbody>
</table>

### Ex. #4 Find \( f'(x) \)

a) \( f(x) = \frac{3}{2}x^2 + 2 \sin x \)  

b) \( f(x) = \frac{-2}{x^7} - 5 \cos x + \frac{8 \sin x}{3} \)
c) \( y = \sqrt{x} + \pi \cos x - 4 \sin x \)  

d) \( g(x) = \frac{3}{2} x^5 - 2x + 1 \)

e) \( y = \sqrt[5]{x^2} + \sqrt{x} - \sqrt{x^{11}} \)  

f) \( h(\theta) = -4\pi \cos \theta + 2 \sin \theta \)
Higher Order Derivatives

We can take successive derivatives of the same function.
If \( f(x) = 2x^3 - 3x + 1 \) then,

1\(^{st}\) Derivative: \( f'(x) = \)

2\(^{nd}\) Derivative: \( f''(x) = \)

3\(^{rd}\) Derivative: \( f'''(x) = \)

4\(^{th}\) Derivative: \( f^{(4)}(x) = \)

5\(^{th}\) Derivative: \( f^{(5)}(x) = \)
Derivative Notation

The previous example showed function notation for derivatives

**Newton Notation**

\[ y = x^4 \]

\[ y' = \]

\[ y'' = \]

\[ y''' = \]

\[ y^{(4)} = \]

**Leibniz Notation**

\[ y = x^{-2} \]

\[ \frac{dy}{dx} = \]

\[ \frac{d^2 y}{dx^2} = \]

\[ \frac{d^3 y}{dx^3} = \]

\[ \frac{d^4 y}{dx^4} = \]
Evaluate

Ex. #6
Given $y = 10x^4$, find $\frac{dy}{dx}$ and $\frac{dy}{dx}\bigg|_{x=2}$. 
Tangent and Normal Lines

**Ex. #4** Given \( f(x) = 3x^2 - 10x \), find the equations of the tangent and normal lines at \( x = 4 \).

**Point on the line:**

**Slope of tangent:**

**Slope of tangent at \( x = 4 \):**

**Slope of normal at \( x = 4 \):**

**Equation of the Tangent Line**

**Equation of the normal Line**
Tangent and Normal Lines

**Ex. #5** Write the equations of the tangent and normal lines at the given point.

a) \( f(x) = 8x^2 - 7; \ x = 3 \)

b) \( f(x) = \frac{1}{x^2}; \ x = 2 \)
2-3 Product and Quotient Rule

HW 2-3
pp.126-127
#1, 5, 16-18, 24, 31, 37, 49, 51, 61, 64, 65, 67, 81, 82, 87, 93, 95, 97, 105-108
Product Rule

Given: \( f(x) = g(x) \cdot h(x) \)

\[ f'(x) = g'(x) \cdot h(x) + h'(x) \cdot g(x) \]

(Derivative of 1st Equation) \cdot (2nd Equation) + (Derivative of 2nd Equation) \cdot (1st Equation)

**Ex. #1**
\[
\begin{align*}
f(x) &= (2x + 3)(x^2 - 7) \\
f'(x) &= \\
\end{align*}
\]

**Ex. #2**
\[
\begin{align*}
f(x) &= 2x^3 + 3x^2 - 14x - 21 \\
f'(x) &= \\
\end{align*}
\]
Quotient Rule

Given \( f(x) = \frac{N(x)}{D(x)} \) then \( f'(x) = \frac{N'(x) \cdot D(x) - D'(x) \cdot N(x)}{[D(x)]^2} \)

Ex. #1 \( f(x) = \frac{x^2 - 1}{x^2 + 1} \)

\( f'(x) = \)
Ex. #4 \[ g(x) = \frac{x}{\cos x} \]

\[ g'(x) = \]
Find the first 3 derivatives of $y = x \cdot \sin x$
The Chain Rule

HW 2-4
pp. 137-139
#7-17 odd, 25, 27, 45, 47, 48, 55, 65, 67, 68, 75, 77, 80a, 81a, 87, 91, 108, 109
Chain Rule

Given: \( y = f[g(x)] \)

Then \( y' = f'[g(x)] \cdot g'(x) \)

Derivative of outer \( \cdot \) Derivative of inner

**Ex. #1** Determine \( \frac{dy}{dx} \) for \( y = (3x^2 - 4)^8 \)

Inner: \( g(x) = 3x^2 - 4 \)

\( g'(x) = \)

Outer: \( f[g(x)] = [g(x)]^8 \)

\( f'[g(x)] = \)
Ex. #2  Determine the 1\textsuperscript{st} Derivative

a) \( y = x^5 \)  

b) \( y = (x^2 + 1)^5 \)

c) \( y = (x^2 + 2x)^3 \)
Ex. #3  Determine the 1st Derivative

a) \( y = 3x^4 \)  

b) \( y = (3x)^4 \)  

C) \( f(x) = \sqrt[3]{(4x - 6)^4} \)
Ex. #4 Determine the 1st Derivative

a) \( f(x) = \sqrt{x^2 - 1} \)  
b) \( y = x^2 \sqrt{5x - 1} \)
## Derivatives of Trigonometric Functions

<table>
<thead>
<tr>
<th>$f(x) = \sin x$</th>
<th>$f(x) = \cos x$</th>
<th>$f(x) = \tan x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x) = \cos x$</td>
<td>$f'(x) = -\sin x$</td>
<td>$f'(x) = \sec^2 x$</td>
</tr>
<tr>
<td>$f(x) = \csc x$</td>
<td>$f(x) = \sec x$</td>
<td>$f(x) = \cot x$</td>
</tr>
<tr>
<td>$f'(x) = -\csc x \cot x$</td>
<td>$f'(x) = \sec x \tan x$</td>
<td>$f'(x) = -\csc^2 x$</td>
</tr>
</tbody>
</table>
Derivatives of Trigonometric Functions & the Chain Rule

<table>
<thead>
<tr>
<th>Derivative</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx}[\sin u] )</td>
<td>( (\cos u) \frac{du}{dx} )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\csc u] )</td>
<td>( (-\csc u \cot u) \frac{du}{dx} )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\cos u] )</td>
<td>( (-\sin u) \frac{du}{dx} )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\sec x] )</td>
<td>( (\sec u \tan u) \frac{du}{dx} )</td>
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</tr>
<tr>
<td>( \frac{d}{dx}[\cot x] )</td>
<td>( (-\csc^2 u) \frac{du}{dx} )</td>
</tr>
</tbody>
</table>
Ex. #5 Determine the 1st Derivative

a) \( y = \cos 2x \)    
b) \( g(x) = 4x^3 - 2\tan 5x \)

c) \( f(\theta) = \cot \pi \theta \)
Ex. #6 Determine the 1st Derivative

a) $h(x) = \sin (2 - 5x)^2$

b) $y = -9\csc^4 (5x)$
Ex. #7 Determine the 1st Derivative

\[ f(x) = \sec^2(x^2) \]
2-5 Implicit Differentiation

**HW 2-5**

pp. 146-147

#1-11 odd, 21, 22, 32, 47, 49, 53, 57
Definition

Implicit differentiation is a type of chain rule used when a function cannot be explicitly solved for \( y \).

Examples of implicit equations:

\[
y x^2 + y^2 = e^y \quad x \sin y - 4 y \sqrt{\cos x} = y^5
\]
Method of Implicit Differentiation

1. Apply the product, quotient, or chain rule to take the derivative of each term.

   Consider \( y \) as a function of \( x \): \( y = f(x) \)

2. Write \( f'(x) \) or \( \frac{dy}{dx} \) whenever you take the derivative of \( y \).

3. Solve for \( f'(x) \) or \( \frac{dy}{dx} \)
EX. #1 Find $\frac{dy}{dx}$ for each.

a) $y = x^3$  

b) $x = y^3$

c) $x^2 - xy + 3y^2 = 7$
EX. #2 \( x^2 + y^2 = 16 \quad \left(3, \sqrt{7}\right) \)

a) Determine \( \frac{dy}{dx} \).

b) Determine the slope of the tangent line at the given point.

c) Write the equation of the tangent line at the given point.
EX. #3  Determine \( \frac{dy}{dx} \) for \[ 5x^2 - xy + y = 12 \]
EX. #4

Determine the first two derivatives of $y^2 - x^2 = 20$
Study Questions for CH.2 TEST

Derivative of a picture

\[ f'(1) = \quad f'(3) = \quad f'(4) = \]

\[ f'(6) = \quad f'(7) = \quad f'(8) = \]

Equation of the tangent line at \( x = 1 \)

Equation of the tangent line at \( x = 4 \)

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How to read Derivative by Definition Problems

When you see these problems, you need to take a derivative of the given equation.

**EX:** \[ \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x \]

Equation: \( y = \sin x \)  
Derivative: \( y' = \cos x \)

**EX#1:** \[ \lim_{h \to 0} \frac{3(x+h)^4 - 3x^4}{h} = \]

**EX#2:** \[ \lim_{h \to 0} \frac{(1+h)^4 - 1}{h} = \]

When you see these problems, you need to take a derivative of the given equation and plug in #.

**EX:** \[ \lim_{h \to 0} \frac{5(2+h)^3 - 40}{h} = 60 \]

Equation: \( y = 5x^3 \)  
Derivative: \( y' = 15x^2 \)

Derivative at \( x = 2 \): \( f'(2) = 60 \)  
Derivative at \( x = \) :
Ex.: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $4x^2 - 5y^2 = 12$

Ex.: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $3x^2 + 2y^2 = 9$

Ex.: Use the given information to find $h'(x)$ and $h'(5)$

\begin{align*}
f(2) &= -1 & f'(2) &= 4 & f(8) &= 3 & f'(8) &= -2 \\
g(2) &= 4 & g'(2) &= 1 & g(8) &= -5 & g'(8) &= -3
\end{align*}

a) $h(x) = f(x) \cdot g(2x^2)$

b) $h(x) = \frac{g(x)}{f(4x)}$

Ex.: Using the information, find the points where the horizontal and vertical tangent(s) occur.

a) $x^2 - 6x + 9 + y^2 - 2y = 0$

and $\frac{dy}{dx} = \frac{3 - x}{y - 1}$

and $\frac{dy}{dx} = \frac{-4x}{3y + 6}$

b) $4x^2 + 3y^2 + 12y = 0$